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# OPTIMAL CONDITIONALLY-RELATIVE OBSERVATION OF NON-STATIONARY LINEAR SYSTEMS†

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A problem of optimal observation arising in control under conditions of uncertainty is considered. The problem is to secure information about the output signal of a dynamical system when the initial state is incompletely defined, by processing incomplete and inaccurate measurements of its states in operation. The problem is investigated for three types of interference in the measuring instrument. Methods for constructing a posteriori and positional solutions are described. The results are illustrated by the example of the observation of a fourth-order dynamical system. 

With the growing complexity of controlled objects and the increased demands made on the quality of control systems, the dimension of the mathematical models used in the theory of controllable systems is increasing steadily. Under such conditions, a priori information about the initial values of many phase variables is frequently known with limited accuracy, and available measuring instruments may measure – inaccurately – only some output signals of the physical system. On the other hand, only some of the phase variables may be of interest for control purposes. When a dynamical system is to be controlled in that situation, one may be obliged to use feedback based on the output rather than feedback based on the state. Since the output signals are measured with limited accuracy, feedback control involves a problem of observation: how to process the accessible measurements in order to derive information about the variables of interest.

In observation theory one uses set-theoretic and probability-theoretic models of uncertainty. Problems of optimal observation with set-theoretic models were first formulated and solved by N. N. Krasovskii [1]. Similar problems were subsequently investigated by others [2–4]. For the stochastic theory of observation (filtering theory) one can consult the monograph edited by Leondes [5].

This paper is related to previous research in which we were involved [6–9]. Its aim is to describe a constructive method for solving the problem of optimal conditionally-relative observation [7–9], taking into account three classes of interference in the measuring instrument. Our main attention will be devoted to an algorithm for the operation of an optimal estimator, which computes in real-time instantaneous estimates of the output signal of a dynamical system. This enables the results to be used for real-time control of systems with uncertainty.

## 1. A POSTERIORI AND POSITIONAL OBSERVATION

Consider a dynamical system, a mathematical model of whose behaviour in a time interval $T = [t_*, t^*]$, $-\infty < t_* < t^* < \infty$ is described by the equation

$$\dot{x} = A(t)x \tag{1.1}$$

where $x = x(t)$ is the $n$-vector (column) of the state of the system at time $t$: $A(t) \in R^{n \times n}$, $t \in T$, is a piecewise-continuous $n \times n$ matrix-valued function.

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We shall assume that the initial state $x(t_*)$ is not known exactly but in the form

$$x(t_*) = x_0 + G\omega$$

where $x_0$ is a known $n$-vector, $G = (g_{(i)}, i \in I = \{1, 2, \dots, n\})$ is an $n \times n_\omega$ matrix ($g_{(i)}$ is its $i$th row) and $\omega$ is an $n_\omega$-vector of unknown parameters belonging to a bounded set

$$\Omega = \{\omega \in R^{n_\omega} : d_* \le \omega \le d^*\}$$

The set $\Omega$ characterizes the a priori uncertainty in the behaviour of system (1.1); we will call it the a priori distribution of parameters of the initial state. Corresponding to it is the a priori distribution $X_0 = x_0 + G\Omega$ of the initial state $x(t_*)$. Note that we are considering here a parallelepiped set only in order to simplify the following arguments; the results may be extended to the case of an arbitrary polyhedral set.

Let us assume that in relation to some control problem we are interested in information about the output signal of system (1.1):

$$z(\tau_*) = Hx(\tau_*)$$

where $\tau_* \in T$ is a given instant of time, $H = (h_j, j \in I)$ is a given $m \times n$ matrix and $h_j$ is its $j$th column. The a priori distribution of the output signal is defined as the set $Z = HX_{\tau_*}$, where $X_{\tau_*} = x_0(\tau_*) + \Phi(\tau_*)\Omega$ is the a priori distribution of the state $x(\tau_*)$ of system (1.1), $x_0(t)$, $t \in T$, is a trajectory of system (1.1) with initial state $x(t_*) = x_0$ and $\Phi(t)$, $t \in T$, is an $n \times n_\omega$ matrix-valued function, which is the solution of the equation

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_*) = G \tag{1.2}$$

If $n_\omega < n$, $m < n$, $\tau_* = t_*$ and the matrices $G$ and $H$ have the form

$$G = (E \in R^{n_\omega \times n_\omega}; 0 \in R^{(n-n_\omega)\times n_\omega})', \quad H = (E \in R^{m \times m}; 0 \in R^{m \times (n-m)})$$

(the prime denotes transposition), then the first $n_\omega$ coordinates of the initial state of system (1.1) are unknown, and the information about the first $m$ coordinates of the initial state is of interest.

To reduce the a priori uncertainty of the vectors $\omega$, $x(t_*)$, $z$, we carry out an observation of system (1.1), processing signals $y(t_*), y(t_* + h), \dots, y(t^*)$ recorded at discrete times $t \in T_h = \{t_*, t_* + h, \dots, t^*\}$ ($h = (t^* - t_*)/N$, where $N$ is a natural number) from a measuring instrument

$$y = c'(t)x + \xi \tag{1.3}$$

where $c(t)$, $t \in T$, is a known continuous $n$-vector function.

The device (1.3) measures, with an error $\xi$, one combination $c_1(t)x_1 + \dots + c_n(t)x_n(t)$ of components of the vector $x$. The measurement-error function $\xi = \xi(t)$, $t \in T$ is piecewise-continuous and satisfies the inequalities

$$\xi_* \le \xi(t) \le \xi^*, \quad t \in T_h \tag{1.4}$$

in which the numbers $\xi_*$, $\xi^*$ characterize the precision of measurement.

The problem of observation is to obtain information about the actually produced output signal $z$ by processing the a priori information $Z$ and the measurement results.

We will distinguish between the problems of a posteriori and positional observation. The problem of a posteriori observation is solved after all measurements have been carried out; the time spent on solving it is immaterial. The problem of positional observation is solved while the measurements are in progress and is aimed at obtaining real-time information about the vector $z$ based on the current measurements. When the positional observation problem is being solved, the time needed to construct estimates of the vector $z$ plays a decisive role. The problems of a posteriori and positional observation are dual analogues of the problems of programmed and positional control.

## 2. OPTIMAL OBSERVATION PROBLEM

Let $y(\cdot) = (y(t), t \in T_h)$ be the collection of all measurements performed.

*Definition*. The set $\hat{\Omega} = \hat{\Omega}(t^*; y(\cdot))$ is called the a posteriori distribution of parameters of the initial state $\omega$ corresponding to a terminal position $(t^*; y(\cdot))$ if it consists of precisely those vectors $\omega \in \Omega$ to which there correspond initial states $x(t_*) = x_0 + G\omega$ that are capable, with certain possible $\xi(t)$, $t \in T_h$, of producing the signal $y(\cdot)$. The elements $\omega \in \hat{\Omega}$ will be called (a posteriori) possible values of the parameters of the initial state.

Corresponding to the set $\hat{\Omega}$ are a posteriori distributions

$$\hat{X}_0 = \hat{X}_0(t^*; y(\cdot)) = x_0 + G\hat{\Omega}, \quad \hat{Z} = \hat{Z}(t^*; y(\cdot)) = H(x_0(\tau_*) + \Phi(\tau_*)\hat{\Omega})$$

of the initial state and the output signal. The set $\hat{X}_0$ was used in [2, 3], but with different names.

For some problems of optimal control with guarantee [10], one is interested not in the entire set $\hat{Z}_0$, but only in certain estimates (numerical characteristics) of it. Accordingly, following the approach in [6], we define the optimal a posteriori observation problem to be the extremal problem ($q$ is a given $m$-vector, $\|q\| = 1$)

$$\hat{\alpha}(t^*; y(\cdot)) = q'z^0(t^*; y(\cdot)) = \max q'z, \quad z \in \hat{Z} \tag{2.1}$$

The vector $z^0 = z^0(t^*; y(\cdot))$ (the extremal output signal) and the corresponding estimate $\hat{\alpha} = \hat{\alpha}(t^*; y(\cdot))$ will be called an a posteriori solution of the optimal observation problem (2.1) (or a solution of the optimal a posteriori observation problem).

We will now formulate the optimal positional observation problem. Suppose $\tau \in T_h$ is an arbitrary instant of time, $y_\tau(\cdot) = (y(t_*), y(t_* + h), \ldots, y(\tau))$ is a sequence of measurements taken up to that time and $Y(\tau)$ is the set of all possible signals $y_\tau(\cdot)$. The set $\hat{\Omega}(\tau) = \hat{\Omega}(\tau, y_\tau(\cdot))$ will be called the actual distribution of parameters of the initial state for the position $(\tau, y_\tau(\cdot))$; corresponding to it are actual distributions of the initial state and the output signal

$$\hat{X}_0(\tau) = \hat{X}_0(\tau; y_\tau(\cdot)) = x_0 + G\hat{\Omega}(\tau)$$

$$\hat{Z}(\tau) = \hat{Z}(\tau; y_\tau(\cdot)) = H(x_0(\tau_*) + \Phi(\tau_*)\hat{\Omega}(\tau))$$

The family of problems

$$\hat{\alpha}(\tau) = \max q'z, \quad z \in \hat{Z}(\tau) \tag{2.2}$$

which depend on the signal $y_\tau(\cdot) \in Y(\tau)$ and the time $\tau \in T_h$ will be called the problem of optimal positional observation.

A solution of this problem (the positional solution of the optimal observation problem (PSOOP)) is defined as the functionals

$$\omega^0(\tau, y_\tau(\cdot)), x^0(\tau, y_\tau(\cdot)), z^0(\tau, y_\tau(\cdot)), \hat{\alpha}(\tau, y_\tau(\cdot)), \quad y_\tau(\cdot) \in Y(\tau), \tau \in T_h \tag{2.3}$$

satisfying the relations

$$x^0(\tau, y_\tau(\cdot)) = x_0 + G\omega^0(\tau, y_\tau(\cdot)), \quad z^0(\tau, y_\tau(\cdot)) = H(x_0(\tau_*) + \Phi(\tau_*)\omega^0(\tau, y_\tau(\cdot)))$$

$$\hat{\alpha}(\tau, y_\tau(\cdot)) = q'z^0(\tau, y_\tau(\cdot)) = \max q'z, \quad z \in \hat{Z}(\tau)$$

Clearly, a PSOOP consists of previously prepared mappings (i.e. before the beginning of the observation process) of all possible measurements onto the estimates of interest. When a PSOOP is known, the observation can be conducted (that is, estimates of the vector $z$ obtained) while the measurements are in progress. To that end it suffices, at each instant of time $\tau \in T_h$, having conducted the next measurement $y(\cdot)$, to construct the vector $y_\tau(\cdot)$ and substitute it into the functionals (2.3), which gives an actual estimate $\hat{\alpha}(\tau, y_\tau(\cdot))$.

PSOOPs are analogues of positional solutions of optimal control problems (optimal feedback control). In control theory, positional solutions using modern measurements of states are implemented through closed-loop control systems, with feedback based on states. PSOOPs are used in control based on incomplete and/or inaccurate measurements and are implemented using part of the closed loop. In the remaining part of the loop the system is controlled on the basis of the estimates obtained in the first part. In combination, both parts of the closed loop implement feedback control based on output. As in the case of optimal feedback control, construction of the PSOOP (2.3) in a closed loop is impossible for non-trivial cases, that is, the principle of optimal observation over a "closed" loop is unrealizable. To implement PSOOPs, therefore, appeal must be made instead to the principle of real-time optimal observation, which is an analogue of the principle of real-time optimal control described in [11]. In real-time optimal observation, the functionals (2.3) are not set up in advance, but the desired estimates are computed in the observation process itself, as measurements are received.

Real-time optimal observation is based on the following analysis.

Suppose a PSOOP (2.3) has been constructed. It is based on the mathematical model (1.1), but is intended, of course, for its physical prototype. Let us assume that the behaviour of the latter is described by an equation

$$\dot{x} = A(t)x + w \tag{2.4}$$

where $w$ is a collection of terms describing the inadequacy of the mathematical model and perturbations affecting the physical system.

Suppose the actual observation process has produced a parameter vector $\omega^*$ of the initial state (unknown to the observer), which generated the initial state $x^*(t_*) = x_0 + G\omega^*$. This initial state of the real physical prototype (2.4), the unknown perturbations $w^*(t), t \in T$, and the errors $\xi^*(t), t \in T_h$ in the work of the measuring instrument generate a trajectory $x^*(t|t_*, x^*(t_*)), t \in [t_*, \tau[$, of system (2.4) and the signal measured up to time $\tau$, say $y^*_\tau(\cdot)$. A solution of problem (2.2) for the position $(\tau, y^*_\tau(\cdot))$ is given by the functionals (2.3) which, along the signal

$$y^*_\tau(t) \equiv c'(t)x^*(t|t_*, x^*(t_*)) + \xi^*(t),\ t \in T_h(\tau) = \{t_*, t_* + h, \dots, \tau\}$$

satisfy the identities

$$x^0(\tau, y^*_\tau(\cdot)) \equiv x_0 + G\omega^0(\tau, y^*_\tau(\cdot)),\quad z^0(\tau, y^*_\tau(\cdot)) \equiv H(x_0(\tau_*) + \Phi(\tau_*)\omega^0(\tau, y^*_\tau(\cdot)))$$

$$\hat{\alpha}(\tau, y^*_\tau(\cdot)) \equiv q'z^0(\tau, y^*_\tau(\cdot)),\quad \tau \in T_h$$

Hence it is obvious that in the actual observation process the PSOOP (2.3) is not used in its entirety (for all $y_\tau(\cdot) \in Y(\tau), \tau \in T_h$); only its values along the signals $y^*_\tau(\cdot), \tau \in T_h$ produced by the measuring instrument are needed.

The functions

$$\omega^*(\tau) = \omega^0(\tau, y^*_\tau(\cdot)),\quad x^*(\tau) = x^0(\tau, y^*_\tau(\cdot))$$

$$z^*(\tau) = z^0(\tau, y^*_\tau(\cdot)),\quad \hat{\alpha}^*(\tau) = \hat{\alpha}(\tau, y^*_\tau(\cdot)), \tau \in T_h$$

will be called a realization of the PSOOP in the specified observation process. A device capable for $\tau \in T_h$ of computing the quantity $z^*(\tau)$ for every actual position $(\tau, y^*_\tau(\cdot))$, in a time $s(\tau)$ not exceeding $h$, will be called an optimal estimator implementing real-time positional observation. Thus, the problem of optimal positional control has been reduced to constructing an algorithm for the operation of an optimal estimator.

## 3. CONSTRUCTION OF AN A POSTERIORI SOLUTION

For an analytical formulation of problem (2.1) we will describe the set $\hat{Z}$. The signal of the measuring instrument (1.3) for a parameter vector of the initial state $\omega \in \Omega$ and error function $\xi(t), t \in T_h$ has the form

$$y(t) = c'(t)x(t) + \xi(t) = c'(t)x_0(t) + c'(t)\Phi(t)\omega + \xi(t),\quad t \in T_h$$

If the measured signal is $y(t), t \in T_h$, then taking into account the restrictions (1.4), $\hat{Z}$ is the set of precisely those vectors $z$ that satisfy the relations

$$z = H(x_0(\tau_*) + \Phi(\tau_*)\omega);$$

$$\xi_* \le y(t) - c'(t)x_0(t) - c'(t)\Phi(t)\omega \le \xi^*,\quad t \in T_h;\quad \omega \in \Omega$$

Put $p' = q'H\Phi(\tau_*)$. Then problem (2.1) becomes

$$p'\omega^0 = \max p'\omega$$
$$\xi_* \le y(t) - c'(t)x_0(t) - c'(t)\Phi(t)\omega \le \xi^*(t),\ t \in T_h;\quad d_* \le \omega \le d^* \tag{3.1}$$

where

$$x^0 = x_0 + G\omega^0,\quad z^0 = H(x_0(\tau_*) + \Phi(\tau_*)\omega^0)$$

Problem (3.1) is an interval linear programming problem with $N + 1$ fundamental constraints and $n_\omega$ variables. Unlike the general linear programming problem, its elements are dynamic in nature. Below we will describe a dual method which is essentially a special rapid dynamic realization of the dual adaptive method of linear programming [12]. As in the case of the optimal control problem, the main time needed to solve optimal observation problems is spent in integrating the direct system; therefore, as in [11], the efficiency of the method is estimated in terms of a basic "time-consumption unit", namely, one integration of the direct system over the entire observation interval $T$.

The main tool of the dual method is the support – a pair

$$K_b = \{T_b, J_b\},\quad T_b \subset T_h,\quad J_b \subset J = \{1, 2, \ldots, n_\omega\},\quad |T_b| = |J_b|$$

such that, if the sets $T_b$ and $J_b$ are non-empty, the matrix

$$D_b = D(T_b, J_b) = (-c'(t)\Phi_j(t),\ j \in J_b,\ t \in T_b)$$

is non-singular. If the sets $T_b$ and $J_b$ are empty, then $K_b$ is empty by definition.

In the solution of problem (3.1) one uses, together with the support $K_b$, certain "associated" elements:

1. A function of potentials $\nu(t)$, $t \in T_h$: $\nu(t) = 0$, $t \in T_n = T_h \backslash T_b$: $\nu_b = (\nu(t),\ t \in T_b)$ – a solution of the equation

$$\nu_b' D_b = p_b',\quad p_b = (p_j,\ j \in J_b) \tag{3.2}$$

2. A vector of estimates

$$\delta' = p' + \sum_{t \in T_b} \nu(t)c'(t)\Phi(t) \tag{3.3}$$

The support components of the vector of estimates are zero: $\delta_j = 0$, $j \in J_b$.

3. A vector of pseudo-parameters $\kappa$; its non-support components $\kappa_j$, $j \in J_n = J \backslash J_b$ are

$$\kappa_j = d_{*j},\ \text{if}\ \delta_j < 0;\quad \kappa_j = d_j^*,\ \text{if}\ \delta_j > 0$$
$$\kappa_j \in [d_{*j}, d_j^*],\ \text{if}\ \delta_j = 0;\quad j \in J_n \tag{3.4}$$

The support components $\kappa_b = (\kappa_j,\ j \in J_b)$ are computed from the equation

$$D_b\kappa_b = (\zeta(t) - y(t) + c'(t)x_0(t) + c'(t)\Phi(t)\kappa_0,\ t \in T_b) \tag{3.5}$$

where

$$\zeta(t) = \xi_*,\ \text{if}\ \nu(t) < 0;\quad \zeta(t) = \xi^*,\ \text{if}\ \nu(t) > 0$$
$$\zeta(t) \in [\xi_*, \xi^*],\ \text{if}\ \nu(t) = 0;\quad t \in T_b$$
$$\kappa_0 = (\kappa_{0j} = 0,\ j \in J_b;\ \kappa_{0j} = \kappa_j,\ j \in J_n)$$

4. A pseudo-error function

$$\zeta(t) = y(t) - c'(t)x_0(t) - c'(t)\Phi(t)\kappa,\quad t \in T_h \tag{3.6}$$

*Definitions.* A support $K_b$ is said to be regular if its associated elements are such that $\nu(t) \ne 0$, $t \in T_b$; $\delta_j \ne 0$, $j \in J_n$.

A support $K_b^0$ is said to be optimal if, for a certain accompanying pseudo-parameter vector $\kappa$ and pseudo-error function $\zeta(t)$, $t \in T_h$, the following inequalities hold

$$d_{*j} \le \kappa_j \le d_j^*, \quad j \in J_b; \quad \xi_* \le \zeta(t) \le \xi^*, \quad t \in T_n$$

A pseudo-parameter vector accompanying an optimal support is a solution of problem (3.1). The extremal output signal is $z^0 = H(x_0(\tau_*) + \Phi(\tau_*)\kappa$.

When the dual method is used to solve an optimal a posteriori observation problem (2.1), the iterations begin with an arbitrary (possibly, empty) support $K_b^1$ and the solution is completed by the construction of an optimal support $K_b^0$. Each iteration of the method consists of replacing the "old" support $K_b$ by a "new" one $\bar{K}_b$, for which the inequality $p'\bar{\kappa} \le p'\kappa$ holds. In what follows, for brevity, only the main operations of the dual method will be described. They are easily justified on the basis of previous results [12].

A time $t \in T_h \backslash \{t_*, t^*\}$ will be called a minimum point of the pseudo-error function $\zeta(t)$, $t \in T_h$ if $\zeta(t) < \zeta(t-h)$ and $\zeta(t) < \zeta(t+h)$; it will be called a maximum point if $\zeta(t) > \zeta(t-h)$ and $\zeta(t) > \zeta(t+h)$. The set of all extremum points will be denoted by $T_0$.

The following information is stored in the computer memory at the start of each iteration: (1) the support $K_b$, (2) the matrix $D_{(b)} = (-c'(t)\Phi(t), t \in T_b)$, (3) the support values of the function of potentials $\nu_b$, (4) the vector of estimates $\delta$, (5) the pseudostate $\kappa$, (6) the set $T_0$, and (7) the quantities $x_0(t)$ and $\Phi(t)$, $t \in T_0 \cup \{t_*, t^*\}$.

Using information 1–7 and formula (3.6), compute

$$\rho^0(K_b) = \max\left\{\rho_{j^0}, \rho(t^0)\right\}$$

$$\rho_{j^0} = \max \rho(\kappa_j, [d_{*j}, d_j^*]), \; j \in J_b$$

$$\rho(t^0) = \max \rho(\zeta(t), [\xi_*, \xi^*]), \; t \in T_0 \cup \{t_*, t^*\}$$

($\rho(c, [a, b])$ is the distance from the number $c$ to the interval $[a, b]$).

If $\rho^0(K_b) = 0$, then $K_b$ is an optimal support. Otherwise, construct the variation of the function of potentials, $\Delta\nu(t)$, $t \in T_h$, depending on the two cases.

*Case* 1. For $\rho^0 = \rho_{j^0}$: $\Delta\nu(t) = 0$, $t \in T_n$; $\Delta\nu_b' D_b = -(\Delta\delta_j, J \in J_b)'$, $\Delta\delta_{j^0} = 1$ if $\kappa_{j^0} > d_{j^0}^*$; $\Delta\delta_{j^0} = -1$ if $\kappa_{j^0} < d_{*j^0}$; $\Delta\delta_j = 0, j \in J_b \backslash t^0$.

*Case* 2. For $\rho^0 = \rho(t^0)$: $\Delta\nu(t) = 0$, $t \in T_n \backslash t^0$; $\Delta\nu_b' D_b = \Delta\nu(t^0)(c'(t^0)\Phi_j(t^0), j \in J_b)'$, $\Delta\nu(t^0) = 1$ if $\zeta(t^0) > \xi^*$; $\Delta\nu(t^0) = -1$ if $\zeta(t^0) < \xi_*$.

Find the variation of the vector of estimates

$$\Delta\delta' = \sum_{t \in T_b \cup t^0} \Delta\nu(t) c'(t) \Phi(t)$$

Now compute

$$\sigma_j = -\delta_j / \Delta\delta_j, \quad \text{if} \quad \delta_j \Delta\delta_j < 0; \quad \sigma_j = \infty, \quad \text{if} \quad \delta_j \Delta\delta_j \ge 0; \quad j \in J_n$$

$$\sigma(t) = -\nu(t)/\Delta\nu(t), \quad \text{if} \quad \nu(t)\Delta\nu(t) < 0$$

$$\sigma(t) = \infty, \quad \text{if} \quad \nu(t)\Delta\nu(t) \ge 0; \quad t \in T_b$$

To simplify the derivations, we shall assume that the numbers $\sigma_j, j \in J_n$ and $\sigma(t)$, $t \in T_b$ are different and non-zero (the general case was investigated in [12]). Number the finite ones in increasing order: $0 < \sigma^1 < \sigma^2 < \ldots < \sigma^{k_0}$.

Compute

$$\alpha^1 = -\rho(K_b); \quad \alpha^{k+1} = \alpha^k + \Delta\alpha^k$$

$$\Delta\alpha^k = (d_{j^k}^* - d_{*j^k})|\Delta\delta_{j^k}|, \quad \text{if} \quad \sigma^k = \sigma_{j^k}$$

$$\Delta\alpha^k = (\xi^* - \xi_*)|\Delta\nu(t^k)|, \quad \text{if} \quad \sigma^k = \sigma(t^k), \quad k = 1, \ldots, k_0$$

A dual step is defined to be a number $\sigma^* = \sigma^{k_*}$ such that $\alpha^{k_*} < 0$, $\alpha^{k_*+1} \geq 0$.

Depending on (a) $\sigma^* = \sigma_{j^{k_*}}$, (b) $\sigma^* = \sigma(t^{k_*})$, construct a new support $\bar{K}_b = \{\bar{T}_b, \bar{J}_b\}$

$$1\text{a}) \ \bar{T}_b = T_b, \quad \bar{J}_b = (J_b \backslash j^0) \cup j^{k_*}; \quad 1\text{b}) \ \bar{T}_b = T_b \backslash t^{k_*}, \quad \bar{J}_b = J_b \backslash j^0$$

$$2\text{a}) \ \bar{T}_b = T_b \cup t^0, \quad \bar{J}_b = J_b \cup j^{k_*}; \quad 2\text{b}) \ \bar{T}_b = (T_b \backslash t^{k_*}) \cup t^0, \quad \bar{J}_b = J_b$$

To complete the iteration, transform the information 2–7 for the new support $\bar{K}_b$.

The matrix $D_{(b)}$ in case 1a is not changed; in case 1b the row of $D_{(b)}$ corresponding to the instant $t^{k_*} \in T_b$ is deleted. In case 2 one adds to $D_{(b)}$ a row $-c'(t^0)\Phi(t^0)$ corresponding to the time $t^0$ (the value of $\Phi(t^0)$ is stored in the computer memory, since $t^0 \in T_0$), and in case 2b, in addition, the row $-c'(t^{k_*})\Phi(t^{k_*})$ is deleted.

The support values of the function of potentials $\bar{\nu}_b$ and the vector of estimates $\bar{\delta}$ are recalculated using formulae (3.2) and (3.3). Rules (3.4) are used to construct the non-support components of the pseudo-parameter vector $\bar{\kappa}_j, j \in J_n$. System (3.5) is solved to find $\bar{\kappa}_b$.

To correct information 6 and 7, construct the variation of the pseudostate $\Delta\kappa = \bar{\kappa} - \kappa$ and the resulting variation of the pseudo-error function $\Delta\zeta(t) = -c'(t)\Phi(t)\Delta\kappa$. Let

$$\zeta(t, \vartheta) = \zeta(t) + \vartheta\Delta\zeta(t), \quad t \in T_h, \ \vartheta \geq 0 \tag{3.7}$$

As $\vartheta$ varies over the interval [0, 1], the extremum points of the function (3.7) will be displaced from extrema $\zeta(t) = \zeta(t, 0)$, $t \in T_h$, to extrema $\bar{\zeta}(t) = \zeta(t, 1)$, $t \in T_h$. The directions $s(t)$, $t \in T_0 \cup \{t_*, t^*\}$, of these displacements are defined as follows: $s(t) = 1$ if $(\zeta(t+h) - \zeta(t))/(\Delta\zeta(t+h) - \Delta\zeta(t)) < 0$, $s(t) = -1$ otherwise, $t \in T_0$; $s(t_*) = 1$, $s(t^*) = -1$.

Accordingly, for each instant of time $t \in T_0 \cup \{t_*, t^*\}$, the following sequence of operations is performed.

1. Compute the number

$$\vartheta(t) = -(\zeta(t + s(t)h) - \zeta(t))/(\Delta\zeta(t + s(t)h) - \Delta\zeta(t)) \tag{3.8}$$

2. If $0 < \vartheta(t) < 1$, then the function (3.7) will have an extremum point $t + s(t)h \in T_h$ in a small right-hand neighbourhood of $\vartheta = \vartheta(t)$. In the set $T_0$, replace the time $t$ by $t + s(t)h$; instead of $x_0(t)$, $\Phi(t)$, store $x_0(t + s(t)h)$ and $\Phi(t + s(t)h)$.

3. Repeat operations 1° and 2° for the time $t + s(t)h$ ($s(t + s(t)h) = s(t)$) until either one of the inequalities $\vartheta(t) \geq 1$, $\vartheta(t) \leq 0$, or the condition $t + s(t)h \notin T_h$ is satisfied. The last two cases characterize the disappearance of the extremum points of the function (3.7); remove the instant $t$ from the set $T_0$ and erase the quantities $x_0(t)$ and $\Phi(t)$ from the computer memory.

*Remark.* In this paper, the case in which new extremum points appear in the the interval $T$ is not considered. It can be investigated using the extremum points of the function $\zeta^1(t) = (\zeta(t+h) - \zeta(t))/h$, $t \in T_h \backslash t^*$.

The time necessary for an iteration of the method is determined by the length of the maximum displacement interval of the extremum points $t \in T_0$. An estimate of its length may be obtained by numerical simulation (see below).

*Theorem* [12]. The method is finite if the supports are regular in the iterations.

A finite modification of the method may be constructed for any problem (3.1) [12].

## 4. IMPLEMENTATION OF A POSITIONAL SOLUTION

According to Section 2, a positional solution is implemented by means of an optimal estimator. Before the beginning of the process, it computes an estimate $\hat{\alpha}^*(t_* - 0)$ and an extremal output signal

$$z^*(t_* - 0) = H(x_0(\tau_*) + \Phi(\tau_*)\omega^*(t^* - 0))$$

using a solution $\omega^*(t_* - 0)$ of the problem

$$p'\omega \to \max, \quad \omega \in \Omega$$

and stores the optimal support $K_b^0\,(t_* - 0)$ for further operations.

Let us assume that the optimal estimator has been working in an interval $[t_*, \tau]$, having computed, using the signal $y^*_\tau(\cdot)$ produced up to then, vectors $\omega^*(t)$, $x^*(t)$, $z^*(t)$, $t \in T_h(\tau)$ and estimates $\hat{\alpha}^*(t)$, $t \in T_h(\tau)$. At time $\tau + h$ the estimator is informed of the measurement $y^*(\tau + h)$ and it must quickly evaluate $\omega^*(\tau + h)$, $x^*(\tau + h)$, $z^*(\tau + h)$, $\hat{\alpha}^*(\tau + h)$.

By assumption, at the previous instant of time $\tau$ (to be precise, in the time interval $[\tau, \tau + s(\tau)[$), the optimal estimator has solved the problem

$$p'\omega \to \max$$
$$\xi_* \le y^*(t) - c'(t)x_0(t) - c'(t)\Phi(t)\omega \le \xi^*, \quad t \in T_h(\tau); \quad d_* \le \omega \le d^* \tag{4.1}$$

and stored its optimal support $K^0_b(\tau)$ and the corresponding data 2–7.

At time $\tau + h$ the estimator solves a problem that differs from (4.1) by the addition of the constraint

$$\xi_* \le y^*(\tau + h) - c'(\tau + h)x_0(\tau + h) - c'(\tau + h)\Phi(\tau + h)\omega \le \xi^* \tag{4.2}$$

To solve problem (4.1), (4.2), the estimator takes as the initial support $K_b(\tau + h)$ an optimal support $K^0_b(\tau)$ of problem (4.1). If the inequalities

$$\xi_* \le \zeta(\tau + h | K_b(\tau + h)) \le \xi^*$$

are satisfied, then $K^0_b(\tau)$ is an optimal support for problem (4.1), (4.2). If not, we have

$$\rho^0(K_b(\tau + h)) = \rho^0(\zeta(\tau + h | K_b(\tau + h)), [\xi_*, \xi^*]) \sim h$$

Hence, a relatively small number of iterations will suffice to correct the initial support $K_b(\tau + h)$ and obtain an optimal support $K^0_b(\tau + h)$, and in the process the extremum points of the pseudo-error function will be slightly displaced. The direct system is iterated over these displacements. Hence it follows that the time needed to solve problem (4.1), (4.2) with initial support $K_b(\tau + h) = K^0_b(\tau)$ is relatively short.

It is obviously impossible to find a formula for estimating the time needed for one correction of the support in the algorithm for the operation of the optimal estimator. Some idea of the efficiency of the algorithm may be derived from the results of a numerical experiment (see the example).

*Remark.* In the case in which the mathematical model (1.1) differs from the physical prototype (2.4) ($w(t) \ne 0$, $t \in T$), the observation process may be broken off at some time $\bar{\tau} \in T_h$ because the constraints of problem (4.1) are incompatible. Something similar happens in control theory when classical optimal feedback is used. Elimination of this effect requires the construction of a theory of optimal observation using non-deterministic mathematical models; this has already been done for optimal control problems. This problem, however, is beyond the scope of the present paper.

*Example.* Let us consider the problem of the observation of a two-mass oscillatory system (1.1) in the time interval $T = [0, 3]$ (Fig. 1)

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2, \quad \dot{x}_4 = 0.1x_1 - x_2 \tag{4.3}$$

Suppose the initial state of system (4.3) is

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = \omega_1, \quad x_4(0) = \omega_2$$

where $\omega = (\omega_1, \omega_2)$ is the parameter vector of the initial state with a priori distribution $\Omega = \{\omega \in R^2: |\omega_i| \le 0.2, i = 1, 2\}$. (It is anticipated that at the initial time $t = 0$ the stationary masses may be subjected to impacts whose intensity cannot exceed known limit values.)

Let us assume that the measuring instrument measures the position of the first mass $x_1(t)$ with an error

$$|\xi(t)| \le 0.15, \quad t \in T_h = \{0, h, 2h, \dots, 3 - h, 3\}, \quad h = 0.03$$

that is, at discrete times $t \in T_h$ the instrument will supply signal values $y(t) = x_1(t) + \xi(t)$. We are interested in the velocity of the second mass at $t = 0$, that is, we will estimate the scalar $z = x_4(0)$. The a priori distribution of the output signal is $Z = \{z \in R: |z| \le 0.2\}$.

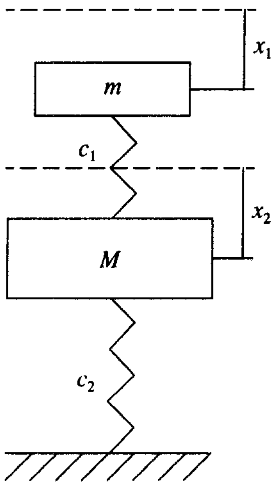


Fig. 1

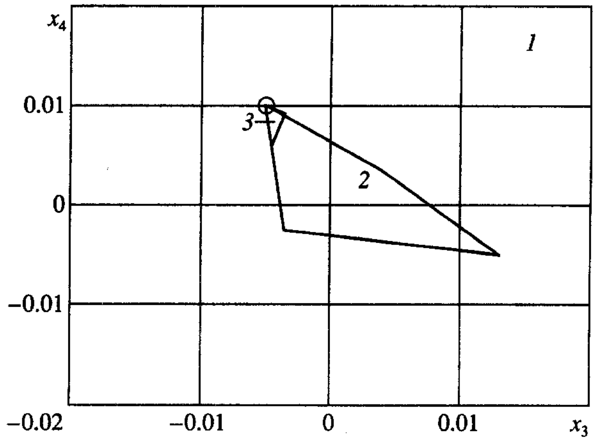


Fig. 2

Let us assume that the model (4.3) is an exact description of the physical system ($w^*(t) \equiv 0, t \in T$), and that the observation process has produced (unknown) values of the components of the initial state

$$x_3^*(0) = -0.05, \quad x_4^*(0) = 0.1 \tag{4.4}$$

and the error function

$$\xi^*(t) = 0.15\sin 2t, \quad t \in [0, 3] \tag{4.5}$$

The solution of the optimal a posteriori observation problem (4.3) produced the following estimate

$$-0.0498241 \le z \le 0.100059$$

If $q = 1$, the time needed to construct an a posteriori solution (the initial support was empty) was found to be 0.34. If $q = -1$, it was 0.36.

Figure 2 illustrates the a priori (set *1*) and a posteriori (set *2*) distributions of the components $x_3(0)$ and $x_4(0)$. In Fig. 3 (curve *1*) we show the behaviour of the estimate $\hat{\alpha}^*(\tau)$, $\tau \in T_h$ for the solution of the positional observation problem with $q = 1$.

While observing system (4.3), having computed the initial state (4.4) and error function (4.5), the following optimal supports $K_b^0(\tau) = \{T_b^0(\tau), J_b^0(\tau)\}$ were obtained

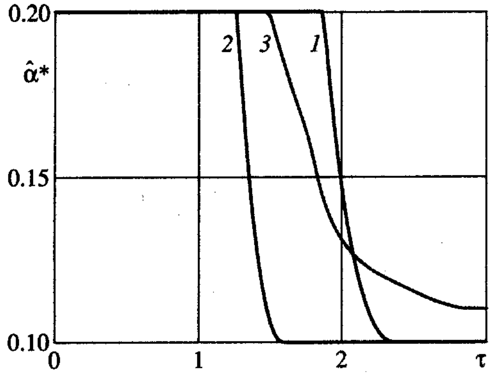


Fig. 3

$$K_b^0(\tau) = \varnothing, \quad \tau \in [0, 1.71[$$

$$T_b^0(\tau) = \{\tau\}, \quad J_b^0(\tau) = \{1\}, \quad \tau = [1.71, 1.89[$$

$$T_b^0(\tau) = \{t_1, \tau\}, \quad J_b^0(\tau) = \{1, 2\}, \quad \tau \in [1.89, 2.4[$$

where $t_1$ is an extremum point of the function $z(t)$, $t \in T_h(\tau)$, having been displaced from $t_1 = 0.69$ to $t_1 = 0.78$; $T_b = \{0.78, 2.37\}, J_b = \{1, 2\}, \tau \in [2{,}4, 3.0]$. Since the optimal support of the problem solved by the optimal estimator has not been changed over the intervals $[0, 1.71[ \cup [2.4, 3.0]$, the time factor needed to correct the data, that is, the length of the interval in which the direct system (1.1) and system (1.2) are integrated to construct $\zeta(\tau + h)$, divided by the length of the observation interval $T$, equals 0.01. Over the interval $[1.71, 2.4[$ the time factor comprises that needed to compute $\zeta(\tau + h)$ (it is equal to 0.01) and that needed to construct the numbers (3.8), which is also 0.01, that is, the total time factor is 0.02. At times $\tau = 1.77$, $\tau = 1.83$, $\tau = 1.95$ the extremum point $t_1$ is displaced by $h$ to the right. Thus, the time consumed at these instants of time is represented by the time factor, which is equal to 0.03. If we let $\gamma$ denote the time necessary for this microprocess for a single integration of system (4.3) over the interval $[0, 3]$, then, if $0.03\gamma < h$, the microprocess may be used to implement the positional solution of the optimal observation problem for system (4.3).

## 5. OPTIMAL OBSERVATION UNDER CONDITIONS OF INERTIAL INTERFERENCE

Let us consider the optimal observation problem formulated in Section 2 on the assumption that

$$n_\omega = m = n, \quad G = E, \quad H = E$$

(the unknown parameter vectors $\omega$ and $z$ are identical with the initial state $x(t_*) = x_0, x_0 \in X_0 = \Omega$).

We will modify the class of functions that describe the measurement errors $\xi(t)$, $t \in T$, by adding, besides inequalities (1.4), the following inequalities

$$\xi_*^1 \le (\xi(t+h) - \xi(t))/h \le \xi^{*1}, \quad t \in T_h \backslash t^*$$

which characterize the inertial property of the error-measurement function.

Let

$$y^1(t) = (y(t+h) - y(t))/h$$

$$(c'\Phi)^1(t) = (c'(t+h)\Phi(t+h) - c'(t)\Phi(t))/h, \quad T_h \backslash t^*$$

Then we obtain the following analytical form of the set $\hat{X}^0$

$$\hat{X}_0 = \{x \in X_0 : \xi_* \le y(t) - c'(t)\Phi(t)x \le \xi^*, \ t \in T_h$$

$$\xi_*^1 \le y^1(t) - (c'\Phi)^1(t)x \le \xi^{*1}, \quad t \in T_h \backslash t^*\}$$

The functional form of the optimal a posteriori observation problem is

$$\hat{\alpha}^1 = p'x^0 = \max p'x, \quad \xi_* - y(t) \le -c'(t)\Phi(t)x \le \xi^* - y(t),\ t \in T_h$$
$$\xi^1_* - y^1(t) \le -(c'\Phi)^1(t)x \le \xi^{*1} - y^1(t),\ t \in T_h \backslash t^*, \quad d_* \le x \le d^* \tag{5.1}$$

We will now describe the main elements of the dual method for solving problem (5.1). A support of problem (5.1) is a triple $K_b = \{T_b, T_b^1, J_b\}$ such that $T_b \subset T_h$, $T_b^1 \subset T_h \backslash t^*$, $J_b \subset J = \{1, 2, \dots, n\}$, $|T_b| + |T_b^1| = |J_b|$ and the matrix $\begin{pmatrix} D_b \\ D_b^1 \end{pmatrix}$ is non-singular, where

$$D_b = (-c'(t)\Phi_j(t),\ j \in J_b,\ t \in T_b), \quad D_b^1 = (-(c'\Phi)^1_j(t), j \in J_b\ t \in T_b^1)$$

The elements associated with the support $K_b$ are as follows.

1. A function of the potentials $\nu(t)$, $t \in T_h$; $\nu^1(t)$, $t \in T_h \backslash t^*$:

$$\nu(t) = 0, \quad t \in T_n = T_h \backslash T_b; \quad \nu^1(t) = 0, \quad t \in T_n^1 = T_h \backslash \{t^*, T_b^1\}$$

$\nu_b = (\nu(t), t \in T_b)$, $\nu_b^1 = (\nu^1(t), t \in T_b^1)$ is a solution of the equation

$$\nu_b' D_b + \nu_b^{1'} D_b^1 = p_b', \quad p_b = (p_j, j \in J_b)$$

2. A vector of the estimates

$$\delta' = p' + \sum_{t \in T_b} \nu(t) c'(t)\Phi(t) + \sum_{t \in T_b^1} \nu^1(t)(c'\Phi)^1(t)$$

3. A pseudostate $\kappa$; the non-support components $\kappa_j$, $j \in J_n$, are given by formula (3.4); the support components $\kappa_b = (\kappa_j, j \in J_b)$ are found from the system of equations

$$D_b \kappa_b = (\zeta(t) - y(t) + c'(t)\Phi(t)\kappa_0,\ t \in T_b)$$
$$D_b^1 \kappa_b = (\zeta^1(t) - y^1(t) + (c'\Phi)^1(t)\kappa_0,\ t \in T_b^1)$$

where

$$\zeta(t) = \xi_*, \quad \text{if} \quad \nu(t) < 0; \quad \zeta(t) = \xi^*, \quad \text{if} \quad \nu(t) > 0$$
$$\zeta(t) \in [\xi_*, \xi^*], \quad \text{if} \quad \nu(t) = 0; \quad t \in T_b$$
$$\zeta^1(t) = \xi^1_*, \quad \text{if} \quad \nu^1(t) < 0; \quad \zeta^1(t) = \xi^{*1}, \quad \text{if} \quad \nu^1(t) > 0$$
$$\zeta^1(t) \in [\xi^1_*, \xi^{*1}], \quad \text{if} \quad \nu^1(t) = 0; \quad t \in T_b^1$$
$$\kappa_0 = (\kappa_{0j} = 0, j \in J_b;\ \kappa_{0j} = \kappa_j, j \in J_n)$$

4. A pseudo-error function

$$\zeta(t) = y(t) - c'(t)\Phi(t)\kappa, \quad t \in T_h; \quad \zeta^1(t) = (\zeta(t+h) - \zeta(t))/h, \quad t \in T_h \backslash t^*$$

*Definition*. A support $K_b^0$ is said to be optimal in problem (5.1) if, for certain associated pseudostates $\kappa$ and pseudo-error functions $\zeta(t)$, $t \in T_h$, $\zeta^1(t)$, $t \in T_h \backslash t^*$, the following inequalities hold

$$d_{*j} \le \kappa_j \le d_j^*, \quad j \in J_b; \quad \xi_* \le \zeta(t) \le \xi^*, \quad t \in T_n; \quad \xi^1_* \le \zeta^1(t) \le \xi^{*1}, \quad t \in T_n^1$$

The pseudostate vector associated with an optimal support is a solution of problem (5.1): $x^0 = \kappa$.

An iteration of the dual method for solving problem (5.1) is constructed by analogy with the algorithm outlined in Section 3, taking into consideration the new definition of the support $K_b$, the additional associated elements, and the rules for constructing them. Note that when the iteration is implemented, one processes not only the extremum points of the pseudo-error function $\zeta(t)$, $t \in T_h$, but also its points of inflection, that is, the extrema of the function $\zeta^1(t)$, $t \in T_h \backslash t^*$.

*Example.* Consider the example of Section 4 with the additional condition that the error function is inertial:

$$\xi(t),\, t \in T_h : |(\xi(t+h) - \xi(t))/h| \le 0.3$$

Note that, in this example, we are considering the optimal conditionally-relative observation problem with inertial interference. Relying on the results of Sections 3 and 5, one can readily construct an algorithm for the a posteriori solution of such problems.

The new a posteriori distribution of the vector $(x_3(0), x_4(0))$ is shown in Fig. 2 (set 3). According to the results obtained by solving the a posteriori observation problem, the output signal $z$ satisfies the inequalities

$$0.0598594 \le z \le 0.100059$$

The time factor needed is 0.44 if $q = 1$ (the initial support is empty); if $q = -1$ it is 0.78.

In Fig. 3 (curve 2) we show the behaviour of the estimate $\hat{\alpha}^*(\tau)$, $\tau \in T_h$ during the positional solution of the problem. The time factor needed to construct $\hat{\alpha}^*(\tau)$ for each $\tau \in T_h$ was at most 0.03 (as in the example with non-inertial interference), since the single point of inflection of the function $\zeta(t)$, $t \in T_h$ appeared at time $\tau = 1.56$ and did not change thereafter during the iterations.

## 6. OPTIMAL OBSERVATION UNDER CONDITIONS OF NOISY FINITELY-PARAMETRIC INTERFERENCE

Let the function $\zeta(t)$, $t \in T_h$, have the form

$$\xi(t) = \sum_{s=1}^{s^*} \xi_s \chi_s(t) + \xi_0(t) \tag{6.1}$$

where $\xi_s$, $s \in S = \{1, 2, \ldots, s^*\}$, are unknown parameters, $\chi_s(t)$, $s \in S$, $t \in T$, are known continuous functions and $\xi_0(t)$, $t \in T$, is an unknown piecewise-continuous function characterizing the interference.

We will assume that the components of the error function obey the constraints

$$\xi_{*s} \le \xi_s \le \xi_s^*,\ s \in S;\quad \xi_{*0} \le \xi_0(t) \le \xi_0^*,\ t \in T_h$$

The functional form of the optimal a posteriori observation problem with error function (6.1) is

$$\begin{gathered} p'x^0 = \max p'x,\quad \xi_{*0} - y(t) \le -c'(t)\Phi(t)x - \sum_{s=1}^{s^*} \xi_s \chi_s(t) \le \xi_0^* - y(t),\quad t \in T_h \\ d_* \le x \le d^*;\quad \xi_{*s} \le \xi_s \le \xi_s^*,\ s \in S \end{gathered} \tag{6.2}$$

A support of problem (6.2) is a triple $K_b = \{T_b, J_b, S_b\}$ ($T_b \subset T_h, J_b \subset J, S_b \subset S$, $|T_b| = |J_b| + |S_b|$) such that the matrix $\|D_b, Q_b\|$ is non-singular, where

$$D_b = (-c'(t)\Phi_j(t), j \in J_b, t \in T_b),\quad Q_b = (-\chi_s(t), s \in S_b, t \in T_b)$$

The support $K_b$ is associated with the following elements.

1. A function of the potentials $\nu(t)$, $t \in T_h$: $\nu(t) = 0$, $t \in T_n = T_h \backslash T_b$; $\nu_b = (\nu(t), t \in T_b)$ – a solution of the equation

$$\nu_b' D_b = p_b',\quad \nu_b' Q_b = 0,\quad p_b = (p_j, j \in J_b)$$

2. A vector of the state estimates

$$\delta' = p' + \sum_{t \in T_b} \nu(t)c'(t)\Phi(t)$$

and a vector of the estimates of the error function parameters

$$\delta_s^{\xi} = \sum_{t \in T_b} \nu(t)\chi_s(t), \quad s \in S$$

3. A pseudostate $\kappa \in R^n$ and a vector of the pseudoparameters $\zeta \in R^{s^*}$. The non-support components $\kappa_j, j \in J_n$, are given by formula (3.4); the non-support components $\zeta_s, s \in S_n = S \backslash S_b$, have the form

$$\zeta_s = \xi_{*s}, \quad \text{if} \quad \delta_s^{\xi} < 0; \quad \zeta_s = \xi_s^*, \quad \text{if} \quad \delta_s^{\xi} > 0; \quad \zeta_s \in [\xi_{*s}, \xi_s^*], \quad \text{if} \quad \delta_s^{\xi} = 0;$$
$$s \in S_n$$

and the support components $\kappa_b$ and $\zeta_b = (\zeta_s, s \in S)$ are computed as a solution of the equation

$$D_b\kappa_b + Q_b\zeta_b = \left(\zeta(t) - y(t) + c'(t)\Phi(t)\kappa_0 + \sum_{s \in S_n} \zeta_s\chi_s(t),\ t \in T_b\right)$$

where

$$\zeta(t) = \xi_{*0}, \quad \text{if} \quad \nu(t) < 0; \quad \zeta(t) = \xi_0^*, \quad \text{if} \quad \nu(t) > 0$$
$$\zeta(t) \in [\xi_{*0}, \xi_0^*], \quad \text{if} \quad \nu(t) = 0; \quad t \in T_b$$

4. A pseudo-error function

$$\zeta(t) = y(t) - c'(t)\Phi(t)\kappa - \sum_{s=1}^{s^*} \zeta_s\chi_s(t), \quad t \in T_h$$

*Definition*. A support $K_b^0$ is said to be optimal in problem (6.2) if, for certain elements $\kappa$, $\zeta_s$, $s \in S$, and $\zeta(t)$, $t \in T_h$, associated with it, the following inequalities hold

$$d_{*j} \le \kappa_j \le d_j^*, \quad j \in J_b; \quad \xi_{*s} \le \xi_s \le \xi_s^*, \quad s \in S_b; \quad \xi_{*0} \le \zeta(t) \le \xi_0^*, \quad t \in T_n$$

A pseudostate vector associated with an optimal support is a solution of problem (6.2): $x^0 = \kappa$.

Subject to these new definitions of the support $K_b$, the optimal support $K_b^0$ and the associated elements, one can construct a dual solution algorithm for problem (6.2), by analogy with the method proposed in Section 3.

As is obvious from the material of Section 4, the algorithm for the operation of an optimal estimator relies on the dual method for constructing an a posteriori solution, using the latter at each stage to correct the optimal support constructed at the preceding stage. Therefore, based on the methods for constructing an a posteriori solution of optimal observation problems under conditions of inertial and noisy finitely-parametric interference, it is not difficult, following the material of Section 4, to describe algorithms for the operation of optimal estimators for these problems.

*Example*. Suppose the possible error functions in the example of Section 4 are

$$\xi(t) = \xi_0(t) + \xi_1 \sin(2t) + \xi_2 \cos(4t); \quad |\xi_0(t)| \le 0.01, \quad |\xi_i| \le 0.15, \quad i = 1, 2$$

During the observation process, the same error function was implemented as in the above examples. Solution of the a posteriori observation problem produced the following estimate for the velocity of the second mass

$$0.086867 \le z \le 0.110098$$

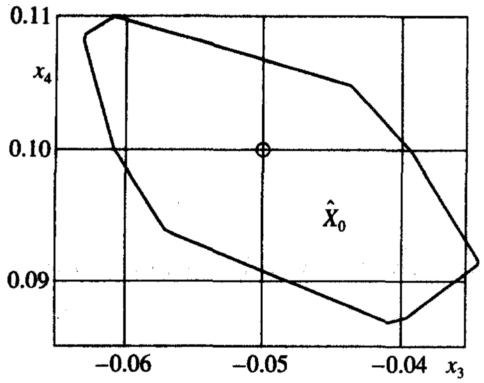


Fig. 4

The time factor needed to construct the solution, for $q = 1$, was equal to 0.86.

Figure 4 illustrates the a posteriori distribution of the components of the initial state $x_3(0)$, $x_4(0)$. Curve 3 in Fig. 3 represents the estimates $\hat{\alpha}^*(\tau)$, $\tau \in T_h$ obtained during the implementation of a positional solution. The time factor needed to correct the supports during the operation of the optimal estimator was at most 0.15.

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*Translated by* D.L.